Analytical Solutions to a Guidance Problem

J. E. Cochran Jr.,* T. S. No,† and D. G. Thaxton‡
Auburn University, Auburn, Alabama 36849

Analytical solutions to the nonlinear equations of relative motion of a constant-velocity target and an interceptor that uses a form of proportional navigation as a guidance law are obtained for navigation gains of 3 and 4. The solutions, in terms of elliptic functions and integrals, are new closed-form solutions to a much investigated guidance problem and, although relatively complicated, may be readily evaluated. An additional result is a condition for intercept that is based on the initial conditions. Numerical results for an exoatmospheric intercept example are included.

Introduction

GUIDANCE of aerospace vehicles involves the use of a variety of guidance laws that, in one way or another, use measured state information to produce commands that the motion of the guided vehicle be changed. One major guidance problem is that of intercepting a target. Although many intercept guidance laws have been proposed, and a large number implemented, proportional navigation[1] (PRONAV), and variations thereof, continue to be the most often used. PRONAV has been used for many years to guide such vehicles.[1]

Because of the practical aspects of intercept guidance laws, numerous investigators have studied and solved kinematic equations that describe the relative motion of targets and interceptors under different simplifying assumptions. In fact, there are several types of proportional navigation laws. To aid in our discussion of some of this previous work and for later reference, we introduce the notation of Fig. 1, where points I and T are used to represent the interceptor and target, respectively. The point O is fixed, and the $OXYZ$ coordinate system is nonrotating. The position vectors $\boldsymbol{R}_I$ and $\boldsymbol{R}_T$ for the interceptor and target, respectively, and the corresponding velocity vectors $\boldsymbol{V}_I$ and $\boldsymbol{V}_T$, can be used to define the relative kinematics since $\boldsymbol{R} = \boldsymbol{R}_T - \boldsymbol{R}_I$ and $\boldsymbol{V} = \boldsymbol{V}_T - \boldsymbol{V}_I$. We can also set

$$\dot{\boldsymbol{V}} = \boldsymbol{A} \tag{1}$$

or

$$\ddot{\boldsymbol{R}} = \boldsymbol{A} \tag{2}$$

The line-of-sight (LOS) unit vector is $\hat{\boldsymbol{u}}_R = \boldsymbol{R}/R$ and, if we denote the angular velocity of the LOS by $\boldsymbol{\Omega}$, then $\mathrm{d}\hat{\boldsymbol{u}}_R/\mathrm{d}t = \boldsymbol{\Omega} x \hat{\boldsymbol{u}}_R$ and $\boldsymbol{V} = \dot{R}\hat{\boldsymbol{u}}_R + \boldsymbol{\Omega} \times \boldsymbol{R}$. Conversely, we may write $\boldsymbol{\Omega} = \boldsymbol{R} \times \boldsymbol{V}/R^2$. Note that if we write $\boldsymbol{\Omega}$ in terms of $\boldsymbol{R}$ and $\boldsymbol{V}$, we implicitly require that

$$\boldsymbol{\Omega} \cdot \hat{\boldsymbol{u}}_R = \Omega_1 = 0$$

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*Professor of Aerospace Engineering. Associate Fellow AIAA.

†Graduate Research Assistant, Department of Aerospace Engineering.

‡Graduate Research Assistant, Department of Aerospace Engineering; currently, Engineer, Sverdrup, Eglin Air Force Base, FL. Member AIAA.

The idea that forms the basis for proportional navigation was probably discovered by ancient seagoing navigators[1] who used it to establish "collision" courses with other vessels. This was done by navigating and guiding so that a constant bearing course was achieved; i.e., the vector corresponding to $\boldsymbol{R}$ in Fig. 1 did not rotate. Then, if the range rate $\dot{R}$ were negative, a collision (intercept) would occur.

Related Investigations

Perhaps the earliest paper to apply this idea to the guidance of missiles is Newell's.[2] Other early investigators include Spits[3] and Bennett and Mathews.[4] Adler[5] derived nonlinear kinematic equations for three-dimensional proportional navigation and also obtained linearized versions of the equations. Murtaugh and Criel[6] considered planar motion in a tutorial article and obtained a solution for the LOS angular velocity assuming that $\dot{R}$ remains constant. Guelman[7] obtained a complicated closed-form solution for the planar motion case under the assumption that $\boldsymbol{A}$ is directed perpendicular to $\boldsymbol{R}$ (true proportional navigation) and has a value equal to a constant c times the LOS rate $\boldsymbol{\Omega}$, i.e., $\boldsymbol{A} = c\Omega\hat{\boldsymbol{u}}_1$, where $\hat{\boldsymbol{u}}_1$ is in the direction opposite to the component of $\boldsymbol{V}$ that is perpendicular to $\boldsymbol{R}$.

Yang et al.[8] developed closed-form solutions to two-dimensional generalized proportional navigation equations in which the commanded acceleration has two components, one perpendicular to and one along the LOS. Yang and Yeh[9] assumed that the magnitudes of these acceleration components were functions of the polar angle between $\boldsymbol{R}$ and a reference direction.

Variable Range-Rate Form of Proportional Navigation

In many applications, the angular motion of the LOS is not small and linearization of the equations is not valid even

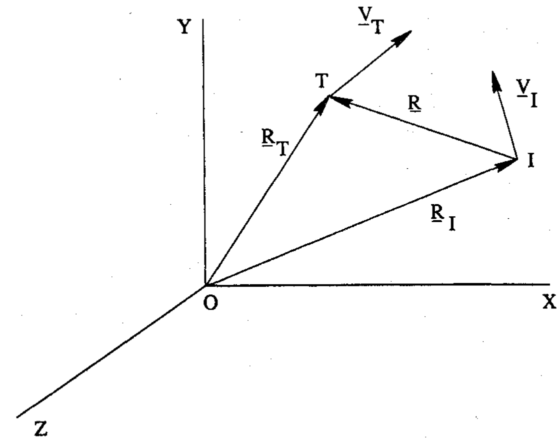


Fig. 1 Engagement geometry.

though the target's velocity may be fairly constant. This is particularly true during the first portion of many engagements. In such cases, even though the applied acceleration is perpendicular to the LOS, the range rate $\dot{R}$ is not constant. However, $\dot{R}$ can be readily measured using a Doppler radar. The commanded relative acceleration then can be taken as

$$\boldsymbol{A} = -K\dot{R}\hat{\boldsymbol{u}}_R \times \boldsymbol{\Omega} \tag{3}$$

where K is a constant to be chosen. If $\dot{R}$ is assumed to be constant, then Eq. (3) yields the guidance law considered by Guelman.[7]

The purpose of this paper is to obtain two analytical solutions to the guidance problem defined by Eqs. (2) and (3) for the cases $K = 3$ and 4, respectively. These solutions are not restricted to small angles, small angular rates, or constant closing speed. They are of both academic and practical interest. First, the solutions are to a problem for which no exact analytical solutions have been published. Second, they provide practical information in the forms of a condition for intercept and analytical acceleration estimates. Regarding the intercept condition, if it is not satisfied, the range rate changes sign from negative to positive for $R > 0$, indicating that an intercept is not possible for the particular set of initial conditions corresponding to the engagement and the value of K used.

Equations of Relative Motion

The equations of relative motion can be expressed in terms of the variables R, $\boldsymbol{\Omega}$, and the three angles, Θ_1, Θ_2, and Θ_3, which are shown in Fig. 2. In Fig. 2, the $IX_IY_IZ_I$ coordinate system translates but does not rotate.

For the relative motion we can write

$$\boldsymbol{R} = R\hat{\boldsymbol{u}}_R \tag{4}$$

The relative velocity can then be obtained as

$$\boldsymbol{V} = \dot{R}\hat{\boldsymbol{u}}_R + \Omega \times \boldsymbol{R} \tag{5}$$

and the relative acceleration is

$$\boldsymbol{A} = \ddot{R}\hat{\boldsymbol{u}}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{R}) + 2\boldsymbol{\Omega} \times \dot{R}\hat{\boldsymbol{u}}_R + \dot{\boldsymbol{\Omega}} \times R \tag{6}$$

Now, $\boldsymbol{\Omega} = \boldsymbol{R} \times \boldsymbol{V}/R^2$ is perpendicular to $\boldsymbol{R}$ [see Eq. (1)], and $\dot{\boldsymbol{\Omega}} = -2(\boldsymbol{R} \times \boldsymbol{V})\dot{R}/R^3 + \boldsymbol{R} \times \boldsymbol{A}/R^2$ is also perpendicular to $\boldsymbol{R}$. In fact, by expanding the cross product of $\boldsymbol{R}$ with Eq. (6) and using the guidance law [Eq. (3)] for $\boldsymbol{A}$, dividing by R^2 and rearranging the result, we get the vector equation

$$\dot{\boldsymbol{\Omega}} + (2-K)\dot{R}/R\boldsymbol{\Omega} = \boldsymbol{0} \tag{7}$$

Additionally, by taking the dot product of $\boldsymbol{R}$ and Eq. (6), we obtain the scalar equation,

$$\ddot{R} - \boldsymbol{\Omega} \cdot \boldsymbol{\Omega} R = 0 \tag{8}$$

Because Ω has no component parallel to $\boldsymbol{R}$, we may orient the coordinate system $Ixyz$ and its associated unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ such that $\hat{\imath} = \hat{\boldsymbol{u}}_R$, $\hat{\jmath}$ is in the direction of $-\boldsymbol{R} \times \boldsymbol{V}$, and $\hat{k}$ completes the right-handed system. Then, we can write $\Omega = \Omega_2\hat{\jmath} + \Omega_3\hat{k}$ and use the angles Θ_3, Θ_2, and Θ_1 in a 3-2-1 sequence to obtain kinematic equations,

$$\dot{\Theta}_1 = (\Omega_2 \sin\Theta_1 + \Omega_3 \cos\Theta_1)\tan\Theta_2 \tag{9a}$$

$$\dot{\Theta}_2 = \Omega_2 \cos\Theta_1 - \Omega_3 \sin\Theta_1 \tag{9b}$$

$$\dot{\Theta}_3 = (\Omega_2 \sin\Theta_1 + \Omega_3 \cos\Theta_1)/\cos\Theta_2 \tag{9c}$$

Equations (7–9) form a complete set of kinematic equations for the relative motion.

Solutions

A solution to Eq. (7) is relatively easy to obtain even for $\dot{R} \neq$ constant. First, for convenience, we introduce the variable τ, defined by

$$\frac{d\tau}{dt} = -\dot{R}/R, \qquad R \neq 0 \tag{10}$$

Or, in integrated form,

$$\tau = -\ell n(R/R_0) \tag{11}$$

where R_0 is the value of R when $\tau = 0$. Then, we can change variables from t to τ to get

$$\frac{d\boldsymbol{\Omega}}{dt} = \left(\frac{d\boldsymbol{\Omega}}{d\tau}\right)\left(\frac{d\tau}{dt}\right) = -\frac{d\boldsymbol{\Omega}}{d\tau}(\dot{R}/R) \tag{12}$$

Thus, Eq. (7) may be replaced by

$$\frac{d\boldsymbol{\Omega}}{d\tau} = (2-K)\boldsymbol{\Omega} \tag{13}$$

The solution to Eq. (13) is

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_0 e^{(2-K)\tau} \tag{14}$$

where $\boldsymbol{\Omega}_0$ is the value of $\boldsymbol{\Omega}$ when $\tau = 0$. Replacing τ with $-\ell n(R/R_0)$ in Eq. (14), we get the simple result,

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_0(R/R_0)^{-(2-K)} \tag{15}$$

Note that Eq. (15) is valid regardless of the form of the variation of R with time. However, to find the time history of $\boldsymbol{\Omega}$, we must solve Eq. (8).

For $\dot{R} < 0$ initially, according to Eq. (15), $|\boldsymbol{\Omega}|$ will initially decrease if we also have $K > 2$. If $|\boldsymbol{\Omega}|$ goes to zero before $\dot{R}$ changes sign, then, from Eq. (8), R will go to zero also. A fundamental question to be answered is, therefore, under what conditions will $\dot{R}$ remain negative? Before we address this question, we note from Eq. (14) that $\boldsymbol{\Omega}$ has a constant direction, i.e., $\boldsymbol{R}$ and $\boldsymbol{V}$ at $t = 0$ define a fixed plane. Hence, we can pick our coordinates so that $\boldsymbol{\Theta}_2$ is always zero and $\boldsymbol{\Theta}_1$ is not needed. This does not necessarily require a constant velocity target, but it does require that the total relative acceleration is given by Eq. (3).

From Eq. (8) and Eq. (15), we have

$$\ddot{R} = (\Omega_0^2 R_0^{4-2K})R^{2K-3} \tag{16}$$

By setting $\rho = R/R_0$, we may rewrite Eq. (16) as

$$\ddot{\rho} = \Omega_0^2 \rho^{2K-3} \tag{17}$$

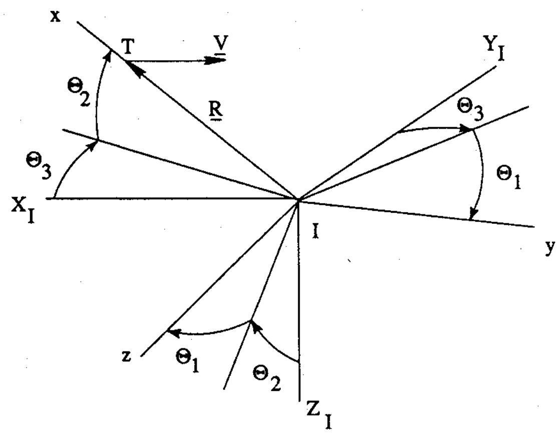


Fig. 2 Relative motion variables.

By multiplying Eq. (17) by $\dot{\rho}$ and integrating the resulting equation with respect to time, we find that

$$\dot{\rho}^2 - \dot{\rho}_0^2 = [\Omega_0^2/(K-1)][\rho^{(2K-2)} - 1] \tag{18}$$

Next, we let

$$\beta_K = \Omega_0\sqrt{(K-1)} \tag{19}$$

$$\alpha_K = \dot{\rho}_0^2/\beta_K^2 - 1 \tag{20}$$

to obtain the more compact equation,

$$\dot{\rho}^2 = \beta_K^2(\rho^{2K-2} + \alpha_K) \tag{21}$$

Note that if $\alpha_K < 0$ and $K > 1$, then $\dot{\rho}$ may change sign when $\rho > 0$ and ρ will not reach 0. Thus, the conditions that must be satisfied for intercept are $\dot{\rho}_0 < 0$ and

$$\dot{\rho}_0^2 > \Omega_0^2/(K-1), \qquad K \neq 1 \tag{22}$$

Inequality (22) is a much simpler condition than that obtained by Guelman[7] for the constant gain case.

It follows from Eq. (21) that

$$\int_0^t \beta_K \,\mathrm{d}t = \pm \int_1^\rho \frac{\mathrm{d}\rho}{\sqrt{\rho^{2K-2} + \alpha_K}} \tag{23}$$

Since $\dot{R}$ must be negative at $t = 0$ for an intercept to occur and $\beta_K > 0$, we chose the negative sign in Eq. (23).

At this point, we must pick a value of K before we can proceed further toward an analytical solution. In general, $K \geq 3$ for conventional proportional navigation applications and, ordinarily, K is chosen to be an integer. Here, we consider only integer values of K. For $K = 2$, the integral on the right-hand side of Eq. (23) is elementary. For $K = 3$ and 4, cases of primary interest because these values are often used in proportional navigation systems, it is an elliptic integral, or can be transformed into one. The cases of $K = 3$ and $K = 4$ are addressed next. For K not an integer we have obtained no analytical solutions.

Case 1: $K = 3$

In this case, we have, from Eq. (23),

$$-\beta_K t = \int_1^\rho \frac{\mathrm{d}\rho}{\sqrt{\rho^4 + \alpha_3}} \tag{24}$$

Since for $\alpha_3 < 0$ intercept will not occur, we consider only the subcase $\alpha_3 > 0$. For convenience, we introduce a constant $B > 0$ such that $B^4 = \alpha_3$. Then, if we define (see Ref. 10, page 64)

$$u = -[\beta_3/(2B)]t + u_0 \tag{25}$$

the solution for ρ^2 is

$$\rho^2 = B^2[(1 + cnu)/(1 - cnu)] \tag{26}$$

where cnu is one of the Jacobian elliptic functions. For this case, these functions have modulus $k = 1/2$. From Eq. (26), we have

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -2B^2\beta_3 \frac{dnu}{1 - cnu} \tag{27}$$

where $dnu = \sqrt{1 - k^2 sn^2 u}$ and snu are the other two Jacobian elliptic functions.

Referring next to Eq. (15) and requiring that the coordinate system that we use has its $\hat{z}$ axis along $\boldsymbol{\Omega}_0$, we can write $\Theta_2 = 0$ and

$$\dot{\Theta}_3 = \Omega_0 \rho \tag{28}$$

Thus, since $\mathrm{d}\Theta_3/\mathrm{d}t = (\mathrm{d}\Theta_3/\mathrm{d}u)(\mathrm{d}u/\mathrm{d}t)$, we have

$$\frac{\mathrm{d}\Theta_3}{\mathrm{d}u} = 2\left(\frac{\Omega_0}{\beta_3}\right)B\sqrt{\frac{1 + cnu}{1 - cnu}} \tag{29}$$

The first identity in 129.04 of Ref. 10, page 25, provides

$$\sqrt{\frac{1 + cnu}{1 - cnu}} = \frac{cn(u/2)/dn^2(u/2)}{sn(u/2)/dn(u/2)} \tag{30}$$

Now, $\mathrm{d}[sn(u/2)/dn(u/2)]/\mathrm{d}u = +(1/2)cn(u/2)/dn^2(u/2)$. Hence,

$$\sqrt{\frac{1 + cnu}{1 - cnu}} = 2\frac{\mathrm{d}[sn(u/2)/dn(u/2)]/\mathrm{d}u}{[sn(u/2)/dn(u/2)]} \tag{31}$$

The solution,

$$\Theta_3 = \Theta_{30} + (2\Omega_0 B/\beta_3)\,\ell n[sn(u/2)/dn(u/2)]\Big|_{u_0}^{u} \tag{32}$$

then follows from Eqs. (29) and (30).

Equations (25–28) and (32) provide the information needed to calculate $\boldsymbol{R}$ and $\boldsymbol{V}$ at any time t, given $\boldsymbol{R}$ and $\boldsymbol{V}$ at $t = 0$.

Case 2: $K = 4$

By setting $K = 4$ in Eq. (23), we get

$$-\beta_4 t = \int_1^\rho \frac{\mathrm{d}\rho}{\sqrt{\rho^6 + \alpha_4}} \tag{33}$$

For $\alpha_4 > 0$, we define $C > 0$ such that $C^6 = \alpha_4$. Next, we introduce a new variable,

$$r \triangleq \rho/C \tag{34}$$

which we can use to rewrite Eq. (33) in the form,

$$-\beta_4 t = \frac{1}{C^2}\int_{1/C}^{\rho/C} \frac{\mathrm{d}r}{\sqrt{1 + r^6}} \tag{35}$$

The integral in Eq. (35) may be cast in the form of a standard elliptic integral. Following Ref. 10, page 258, we let $z = r^2$ so that

$$\mathrm{d}r = \frac{\mathrm{d}z}{2\sqrt{z}}$$

to get

$$-\beta_4 t = \frac{1}{2C^2}\int_{1/C^2}^{\rho^2/C^2} \frac{\mathrm{d}z}{\sqrt{z(1 + z^3)}} \tag{36}$$

The new form of integral in Eq. (36) may be evaluated using Ref. 10, formula 260.50. The result for ρ^2 is

$$\rho^2 = C^2[(1 - cnu)/(1 + \alpha cnu)]/(\sqrt{3} - 1) \tag{37}$$

where,

$$u = -2C^2\sqrt[4]{3}\beta_4 t + u_0 \tag{38}$$

$$\alpha = (\sqrt{3} + 1)^2/2 \tag{39}$$

and the modulus k of the elliptic functions is given by

$$k^2 = (\sqrt{3} + 2)/4$$

The dimensionless range-rate $\dot{\rho}$ can be found from Eq. (37), viz.,

$$\dot{\rho} = -[2C^3\beta_4\sqrt[4]{27}/(\sqrt{3} - 1)^{3/2}]\, snu\, dnu/ [(1 - cnu)^{1/2}(1 + \alpha cnu)^{3/2}] \tag{40}$$

The time rate of change of Θ_3, formed by using Eqs. (9), (15), and (37), can be used to get the result,

$$\Theta_3 = \Theta_{30} - D \int_{u_0}^{u} \frac{1 - cnu}{1 + \alpha cnu}\, du \tag{41}$$

where $D = 1/[(2\beta_4\sqrt[4]{3})(\sqrt{3} - 1)]$.

The integral in Eq. (41) can be expressed as an elliptic integral of the third kind, plus additional functions of elliptic functions. The solution for Θ_3 is, therefore, fairly involved, but by following Ref. 10, pages 215 and 216, a form suitable for numerical evaluation can be obtained.

Comments on the Analytical Results

The solutions presented are of academic interest and provide some practical insight into guidance problems, especially those involving space vehicles that do not experience changes in their velocities due to atmospheric drag. For example, the equations solved are good representations of engagements of satellites by maneuvering interceptor vehicles that thrust only perpendicularly to the interceptor-to-satellite LOS.

Two important conclusions that can be drawn from the solutions are as follows:

1) If the LOS rate at $t = t_0$ is so large that inequality (22) is not satisfied, intercept will not occur.

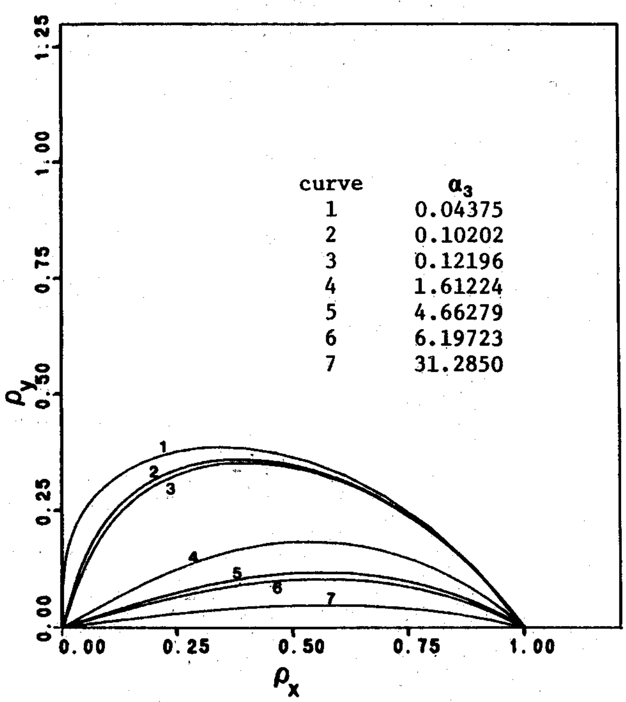


Fig. 3 Target trajectories as seen from the interceptor, $K = 3$.

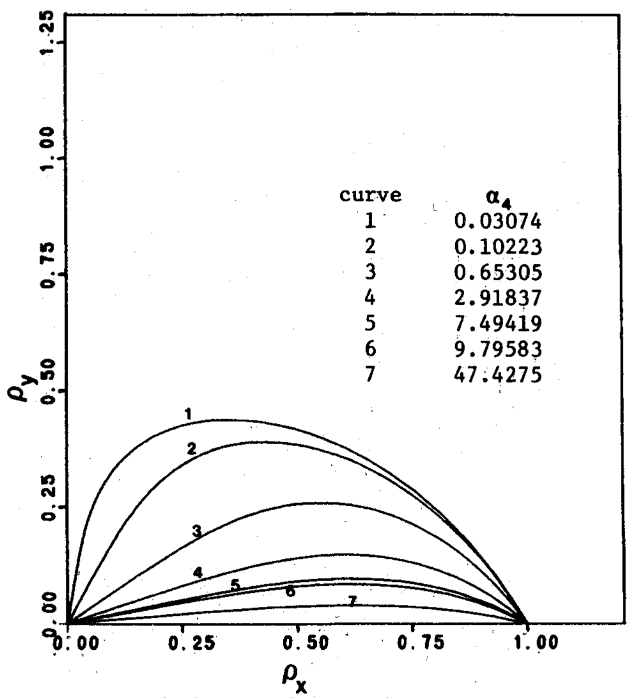


Fig. 4 Target trajectories as seen from the interceptor, $K = 4$.

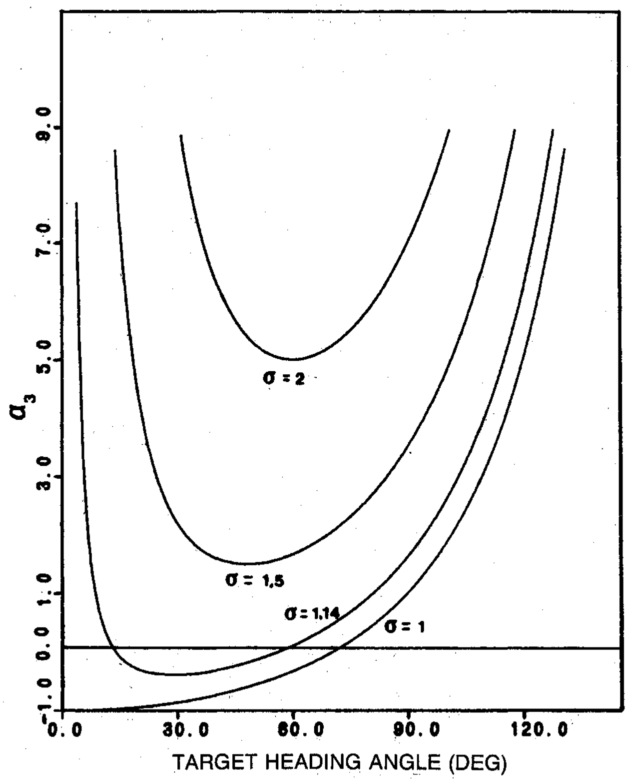


Fig. 5 Parameter α_3 as a function of target heading angle.

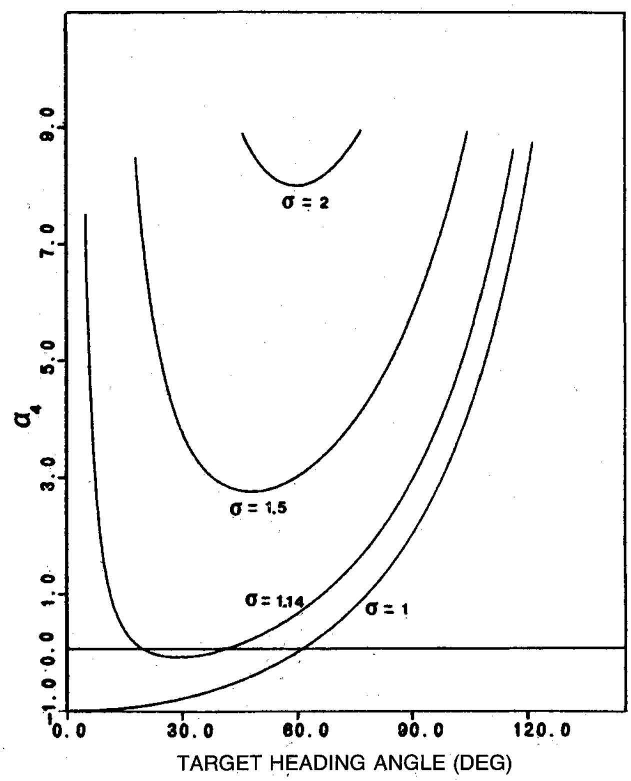
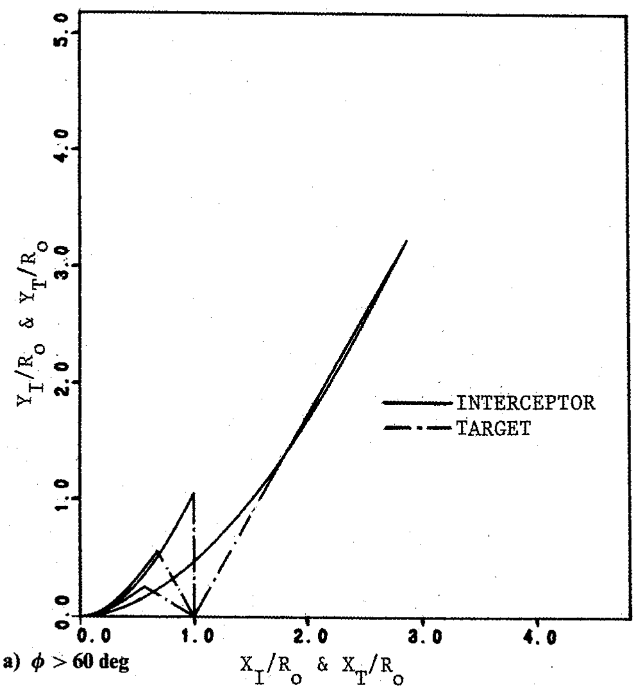
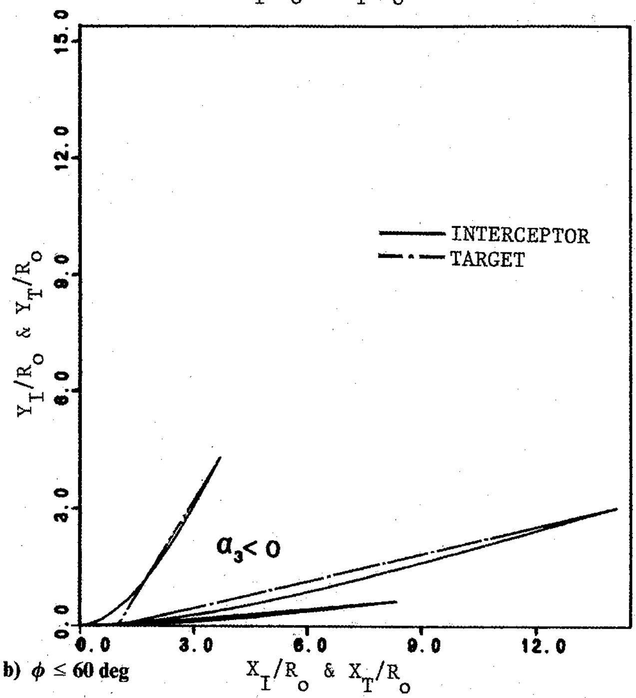


Fig. 6 Parameter α_4 as a function of target heading angle.

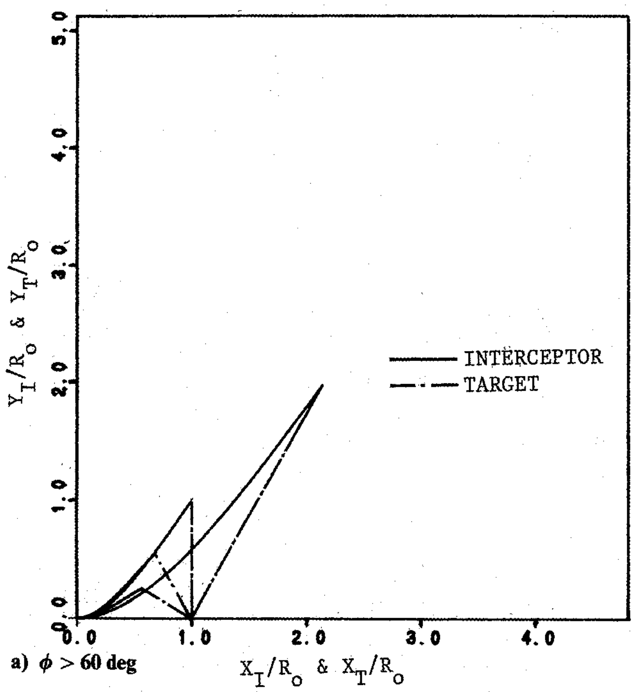


a) $\phi > 60$ deg

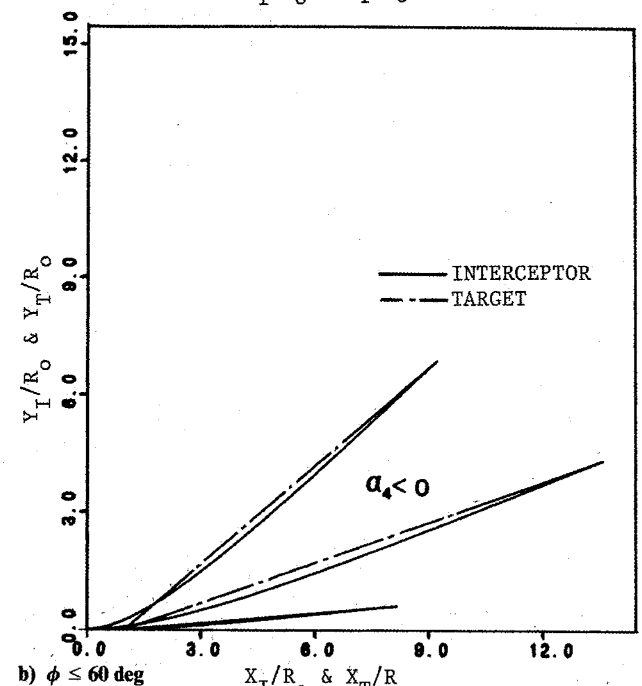


b) $\phi \leq 60$ deg

Fig. 7 Interceptor trajectories for various target heading angles, $K = 3$.



a) $\phi > 60$ deg



b) $\phi \leq 60$ deg

Fig. 8 Interceptor trajectories for various target heading angles, $K = 4$.

2) Since the acceleration magnitude may be written by using Eqs. (3), (15), and (17) as

$$|A| = K|\dot{R}||\Omega| = K\dot{\Theta}_{30} R_0 \dot{\rho} \rho^{(K-2)} \quad (42)$$

the solutions for ρ and $\dot{\rho}$ [Eqs. (26), (27), (36), and (40)] may be used to find explicit analytical expressions for the commanded acceleration if one wishes to work through the details.

Numerical Results

The solutions for $K = 3$ and $K = 4$ have been verified by comparing them with particular solutions generated via numerical integration. A standard fourth-order Runge-Kutta algorithm was used for the numerical solutions. Exact agreement within the limits of accuracy of the computer and numerical integration was achieved.

The analytical solutions [Eqs. (26) and (32) for $K = 3$ and Eqs. (37) and (41) for $K = 4$] may be used to generate trajectories of the target as seen from the interceptor. If we take $\Theta_{30} = 0$, then the parameters that may be varied are $\dot{\rho}_0$ and Ω_0. We may assume that the interceptor is on the X_I axis and traveling in the positive X_I direction at time $t = 0$, and if the target's velocity vector makes an angle ϕ with the positive X_I axis, then $\dot{\rho}_0 = (V_T \cos\phi - V_I)/R_0$ and $\Omega_0 = V_T \sin\phi / R_0$. The important parameter [see Eq. (20)] α_K can be found from,

$$\alpha_K = (K-1)[(\cos\phi - \sigma)/\sin\phi]^2 - 1 \quad (43)$$

where $\sigma = V_I/V_T$. Clearly, for certain values of $\sigma = V_I/V_T$ and ϕ, α_K will be negative. As mentioned earlier, this means an intercept will not occur. The $\alpha_K = 0$ boundaries are defined by the values of ϕ that satisfy the following conditions:

$$\cos\phi_{1,2} = \sigma(K-1)/K \pm (1/K)\sqrt{(1-K)\sigma^2 + K} \quad (44)$$

for values of σ that yield real values of $\cos\phi$.

Example trajectories are shown in Fig. 3 for ($K = 3$), and Fig. 4 for ($K = 4$), where without loss of generality the target always starts on the $\rho_{\hat{x}}$ axis at 1. Plots of α_3 and α_4 vs ϕ for several values of the ratio V_I/V_T are given in Figs. 5 and 6, respectively. For initial conditions that correspond to $\alpha_K < 0$, $\dot{R}$ goes to zero before $\boldsymbol{\Omega} = 0$ and a "miss" occurs.

Typical spacecraft intercept scenarios are illustrated by the trajectories (in inertial space) shown in Figs. 7 and 8. The basic engagement conditions are $V_I = 8$ km/s, $V_T = 7$ km/s, $R_0 = 200$ km, and $K = 3$ and 4, respectively. The initial direction of the target's velocity was varied to obtain several trajectories for each K value. Note that for initial conditions such that $\alpha_K < 0$, no trajectories are shown since they all miss the target. Figures 7b and 8b illustrate the ranges of heading angles for which intercept will not occur.

It should be noted that the guidance law [Eq. (3)] requires that, when $\dot{R}$ becomes small, the commanded acceleration also becomes small for bounded LOS rate. Further, when $\dot{R}$ changes sign, the sign of the commanded acceleration changes sign so that the acceleration increases $|\boldsymbol{\Omega}|$ instead of decreasing it.

Conclusions

Analytical solutions to the equations that govern the relative motion of point mass models of a constant-velocity target and an interceptor that uses a form of proportional navigation as a guidance law have been obtained for navigation gains of 3 and 4. Although fairly complicated, these new solutions may be readily evaluated and provide some additional insight into the general exoatmospheric guidance problem. In particular, in the course of obtaining the solutions, a condition that must be satisfied if intercept is to be possible was obtained. An exoatmospheric intercept example was used to illustrate the use of the solutions and the intercept condition.

Acknowledgments

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